on obtient

$$\int_{p-1}^{p} \frac{1}{\sqrt{x}}\,dx \geqslant \int_{p-1}^{p} \frac{1}{\sqrt{p}}\,dx$$

$$\int_{p-1}^{p} \frac{dx}{\sqrt{x}} \geqslant \frac{1}{\sqrt{p}}\,(p-p+1)$$

donc

$$\forall p \geqslant 2, \quad \frac{1}{\sqrt{p}} \leqslant \int_{p-1}^{p} \frac{dx}{\sqrt{x}}$$

2 En déduire que pour tout $n \in \mathbb{N}^*$, on a: $-2+2\sqrt{n+1} \leq u_n \leq -1+2\sqrt{n}$

on a $$u_n = \sum_{p=1}^{n} \frac{1}{\sqrt{p}}.$$

on a $$\frac{1}{\sqrt{p}} \leqslant \int_{p-1}^{p} \frac{1}{\sqrt{x}}\,dx \quad \forall p \geqslant 2.$$

$$\Rightarrow \sum_{p=2}^{n} \frac{1}{\sqrt{p}} \leqslant \sum_{p=2}^{n} \int_{p-1}^{p} \frac{1}{\sqrt{x}}\,dx$$

$$u_n - 1 \leqslant \int_{1}^{2} \frac{1}{\sqrt{x}}\,dx + \int_{2}^{3} \frac{dx}{\sqrt{x}} + \cdots + \int_{n-1}^{n} \frac{dx}{\sqrt{x}}$$